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## How tall is the automorphism tower of a group?

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The automorphism tower of a group is obtained by computing its automorphism group, the automorphism group of *that* group, and so on, iterating transfinitely. Each group maps canonically into the next using inner automorphisms, and so at limit stages one can take a direct limit and continue the iteration.

$$G \rightarrow \text{Aut}(G) \rightarrow \text{Aut}(\text{Aut}(G)) \rightarrow \cdots \rightarrow G_\omega \rightarrow G_{\omega+1} \rightarrow \cdots \rightarrow G_\alpha \rightarrow \cdots$$

The tower is said to terminate if a fixed point is reached, that is, if a group is reached which is isomorphic to its automorphism group by the natural map. This occurs if a *complete* group is reached, one which is centerless and has only inner automorphisms.

The natural map  $\pi : G \rightarrow \text{Aut}(G)$  is the one that takes any element  $g \in G$  to the inner automorphism  $i_g$ , defined by simple conjugation  $i_g(h) = ghg^{-1}$ . Thus, the kernel of  $\pi$  is precisely the center of  $G$ , the set of elements which commute with everything in  $G$ , and the range of  $\pi$  is precisely the set of inner automorphisms of  $G$ . By composing the natural maps at every step, one obtains a commuting system of homomorphisms  $\pi_{\alpha,\beta} : G_\alpha \rightarrow G_\beta$  for  $\alpha < \beta$ , and these are the maps which are used to compute the direct limit at limit stages.

Much of the historical analysis of the automorphism tower has focused on the special case of centerless groups, for when the initial group is centerless, matters simplify considerably\*. An easy computation shows that  $\theta \circ i_g \circ \theta = i_{\theta(g)}$  for any automorphism  $\theta$ , and from this we conclude that  $\text{Inn}(G) \triangleleft \text{Aut}(G)$  and, for centerless  $G$ , that  $C_{\text{Aut}(G)}(\text{Inn}(G)) = 1$ . In particular, if  $G$  is centerless then so also is

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\* In Hulse [1970], Rae and Roseblade [1970], and Thomas [1985], the tower is only defined in this special case; but the definition I give here works perfectly well whether or not the group  $G$  is centerless. Of course, when there is a center, one has homomorphisms rather than embeddings.

$\text{Aut}(G)$ , and more generally, by transfinite induction every group in the automorphism tower of a centerless group is centerless. In this case, consequently, all the natural maps  $\pi_{\alpha,\beta}$  are injective, and so by identifying every group with its image under the canonical map, we may view the tower as building upwards to larger and larger groups; the question is whether this building process ever stops.

$$G \subseteq G_1 \subseteq \cdots G_\omega \subseteq \cdots \subseteq G_\alpha \subseteq \cdots$$

In this centerless case, the outer automorphisms of every group become inner automorphisms in the next group. One wants to know, then, whether this process eventually closes off.

The classical result is the following theorem of Wielandt.

**Classical Theorem.** (WIELANDT, 1939) *The automorphism tower of any centerless finite group terminates in finitely many steps.*

Wielandt's theorem, pointed to with admiration at the conclusion of Scott's [1964] book *Group Theory*, was the inspiration for a line of gradual generalizations by various mathematicians over the succeeding decades. Scott closes his book with questions concerning the automorphism tower, specifically mentioning the possibility of transfinite iterations and towers of non-centerless groups.

**Question.** (SCOTT, 1964) *Is there a group whose automorphism tower never terminates?*

By the 1970's, several authors had made progress:

**Theorem.** (RAE AND ROSEBLADE, 1970) *The automorphism tower of any centerless Černikov group terminates in finitely many steps.*

**Theorem.** (HULSE 1970) *The automorphism tower of any centerless polycyclic group terminates in a countable ordinal number of steps.*

These results culminated in Simon Thomas' elegant solution to the automorphism tower problem in the case of centerless groups. An application of Fodor's lemma lies at the heart of Thomas' proof.

**Theorem.** (THOMAS, 1985) *The automorphism tower of any centerless group eventually terminates. Indeed, the automorphism tower of a centerless group  $G$  terminates in fewer than  $(2^{|G|})^+$  many steps.*

In the general case, however, the question remained open whether every group has a terminating automorphism tower. This is settled by the following theorem.

**Main Theorem.** (HAMKINS [1998]) *Every group has a terminating automorphism tower.*

**Proof:** Suppose  $G$  is a group. The following transfinite recursion defines the automorphism tower of  $G$ :

$$\begin{aligned} G_0 &= G \\ G_{\alpha+1} &= \text{Aut}(G_\alpha), \quad \text{where } \pi_{\alpha,\alpha+1} : G_\alpha \rightarrow G_{\alpha+1} \text{ is the natural map,} \\ G_\lambda &= \text{dir lim}_{\alpha < \lambda} G_\alpha, \quad \text{if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

When  $\alpha < \beta$  one obtains the map  $\pi_{\alpha,\beta} : G_\alpha \rightarrow G_\beta$  by composing the canonical maps at each step, and these are the maps used to compute the direct limit at limit stages. Thus, when  $\lambda$  is a limit ordinal, every element of  $G_\lambda$  is of the form  $\pi_{\alpha,\lambda}(g)$  for some  $\alpha < \lambda$  and some  $g \in G_\alpha$ .

Since Simon Thomas [1985] has proved that every centerless group has a terminating automorphism tower, it suffices to show that there is an ordinal  $\gamma$  such that  $G_\gamma$  has a trivial center. For each ordinal  $\alpha$ , let  $H_\alpha = \{g \in G_\alpha \mid \exists \beta \pi_{\alpha,\beta}(g) = 1\}$ . For every  $g \in H_\alpha$  there is some least  $\beta_g > \alpha$  such that  $\pi_{\alpha,\beta_g}(g) = 1$ . Let  $f(\alpha) = \sup_{g \in H_\alpha} \beta_g$ . It is easy to check that if  $\alpha < \beta$  then  $\alpha < f(\alpha) \leq f(\beta)$ . Iterating the function, define  $\gamma_0 = 0$  and  $\gamma_{n+1} = f(\gamma_n)$ . This produces a strictly increasing  $\omega$ -sequence of ordinals whose supremum  $\gamma = \sup\{\gamma_n \mid n \in \omega\}$  is a limit ordinal which is closed under  $f$ . That is,  $f(\alpha) < \gamma$  for every  $\alpha < \gamma$ . I claim that  $G_\gamma$  has a trivial center. To see this, suppose  $g$  is in the center of  $G_\gamma$ . Thus,  $\pi_{\gamma,\gamma+1}(g) = 1$ . Moreover, since  $\gamma$  is a limit ordinal, there is  $\alpha < \gamma$  and  $h \in G_\alpha$  such that  $g = \pi_{\alpha,\gamma}(h)$ . Combining these facts, observe that

$$\pi_{\alpha,\gamma+1}(h) = \pi_{\gamma,\gamma+1}(\pi_{\alpha,\gamma}(h)) = \pi_{\gamma,\gamma+1}(g) = 1.$$

Consequently,  $\pi_{\alpha,f(\alpha)}(h) = 1$ . Since  $f(\alpha) < \gamma$ , it follows that

$$g = \pi_{\alpha,\gamma}(h) = \pi_{f(\alpha),\gamma}(\pi_{\alpha,f(\alpha)}(h)) = \pi_{f(\alpha),\gamma}(1) = 1,$$

as desired.  $\square_{\text{Theorem}}$

So now we know that the automorphism tower of any group terminates. But how long does it take? In the centerless case, Simon Thomas provided an attractive bound on the height of the automorphism tower, namely, the tower of a centerless group  $G$  terminates before  $(2^{|G|})^+$ . It is therefore natural to ask the question for groups in general:

**Question.** *How tall is the automorphism tower of a group  $G$ ?*

Unfortunately, the proof of the Main Theorem above does not reveal exactly how long the automorphism tower takes to stabilize, since it is not clear how large  $f(\alpha)$  can be. Nevertheless, something more can be said. Certainly the automorphism tower of  $G$  terminates well before the next inaccessible cardinal above  $|G|$ . More generally, if  $\lambda > \omega$  is regular and  $|G_\alpha| < \lambda$  whenever  $\alpha < \lambda$ , then I claim the centerless groups will appear before  $\lambda$ . To see this, let  $H_\alpha = \{g \in G_\alpha \mid \exists \beta_g < \lambda \pi_{\alpha, \beta_g}(g) = 1\}$  and define  $f : \lambda \rightarrow \lambda$  by  $f(\alpha) = \sup_{g \in H_\alpha} \beta_g$ ; if  $\gamma < \lambda$  is closed under  $f$ , then it follows as in the main theorem that  $G_\gamma$  has no center, as desired. In this case the tower therefore terminates in fewer than  $(2^{|G|})^+$  many additional steps. If it happens that  $|G_\gamma|^+ < \lambda$ , one can adapt Thomas' [1996] argument using Fodor's lemma to prove that the tower terminates actually in fewer than  $\lambda$  many steps. The point is that one can find a bound on the height of the tower by bounding the rate of growth of the groups in the tower.

Thomas [1985] provides his explicit bound in the case of centerless  $G$  in precisely this way. He proves that if  $G$  is centerless and  $\lambda = (2^{|G|})^+$ , then  $|G_\alpha| < \lambda$  for all  $\alpha < \lambda$ . The analogous result, unfortunately, does not hold for groups with nontrivial centers. This is illustrated by the following example, provided by the anonymous referee of my paper [1998]:

**Example.** *There exists a countable group  $G$  such that  $|\text{Aut } G| = 2^\omega$  and  $|\text{Aut}(\text{Aut } G)| = 2^{2^\omega}$ .*

**Proof:** For each prime  $p$ , let  $\mathbb{Z}[1/p] = \{m/p^n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$  be the additive group of  $p$ -adic rationals and let  $G = \bigoplus_p \mathbb{Z}[1/p]$  be the direct sum of these groups. An element  $g \in G$  is divisible by  $p^n$  for all  $n \in \mathbb{N}$  iff  $g \in \mathbb{Z}[1/p]$ . Hence, if  $\pi \in \text{Aut } G$ , then  $\pi[\mathbb{Z}[1/p]] = \mathbb{Z}[1/p]$  for each prime  $p$ . It is easy to see that any automorphism of  $\mathbb{Z}[1/p]$  is simply multiplication by an element  $u \in U_p = \{\pm p^n \mid n \in \mathbb{Z}\}$ , the group of multiplicative units of the ring of  $p$ -adic rationals. Thus,  $\text{Aut } G \cong \prod_p U_p$ ; and so  $|\text{Aut } G| = 2^\omega$ .

Next note that  $U_p \cong \mathbb{Z} \times C_2$  for each prime  $p$ . Thus  $\text{Aut } G \cong P \times V$ , where  $P$  is the direct product of countably many copies of  $\mathbb{Z}$  and  $V$  is the direct product of countably many copies of  $C_2$ . Since each nonzero element of  $V$  has order 2, it follows that  $V$  is isomorphic to a direct sum of  $|V|$  copies of  $C_2$ . Thus we can identify  $V$  with a vector space of dimension  $2^\omega$  over the field of two elements. Hence,

$$\text{Aut}(\text{Aut } G) \cong \text{Aut } P \times \text{Aut } V = \text{Aut } P \times GL(V),$$

where  $GL(V)$  is the general linear group on the vector space  $V$ . Since  $|GL(V)| = 2^{2^\omega}$ , it follows that  $|\text{Aut}(\text{Aut } G)| = 2^{2^\omega}$ .  $\square$

Enriqueta Rodríguez-Carrington has observed that the natural modification of my argument shows that the derivation tower of every Lie algebra eventually leads to a centerless Lie algebra. Since Simon Thomas [1985] proved that the derivation tower of every centerless Lie algebra must eventually terminate, it follows that the derivation tower of any Lie algebra must eventually terminate.

One might hope, since every step of the automorphism tower kills the center of the previous group, that  $G_\omega$  is always centerless; but this is not so. The dihedral group with eight elements has a center of size two, but is isomorphic to its own automorphism group (there is an outer automorphism which swaps  $a$  and  $b$  in the presentation  $\langle a, b \mid a^2 = 1, b^2 = 1, (ab)^4 = 1 \rangle$ ). The group at stage  $\omega$  is just the two element group, which still has a center, and so this tower survives until  $\omega + 1$ . Simon Thomas has constructed examples showing that for every natural number  $n$  there are finite groups whose tower has height  $\omega + n$ , but these also become centerless at stage  $\omega + 1$ . Perhaps our attention should focus, for an arbitrary group  $G$ , on the least ordinal stage  $\gamma$  such that  $G_\gamma$  is centerless. The main point, then, is to find an explicit bound on how large  $\gamma$  can be in comparison with  $|G|$ .

Let me now ask an innocent question:

**Question.** *Can you predict the height of the automorphism tower of a group  $G$  by looking at  $G$ ?*

Of course, you may argue philosophically, the answer is Yes, because the automorphism tower of a group  $G$  is completely determined by  $G$ ; one simply iterates the automorphism group operation until the termination point is obtained, and that is where the tower terminates. But nevertheless, I counter philosophically that the answer to the question is No! How can this be?

The reason for my negative answer is that the automorphism tower of a group has a set-theoretic essence; building the automorphism tower by iteratively computing automorphism groups is rather like building the Levy hierarchy  $V_\alpha$  by iteratively computing power sets. The fact is that the automorphism tower of a group can depend on the model of set theory in which you compute it, with the very same group leading to wildly different automorphism towers in different set theoretic universes. And so, in order to predict the height of the tower of  $G$ , you can't just look at  $G$ ; you must also look at  $\text{Aut}(G)$  and  $\text{Aut}(\text{Aut}(G))$  and so on; and these depend on the set-theoretic background.

**Theorem.** (HAMKINS AND THOMAS 1997) *It is relatively consistent that for every  $\lambda$  and for every  $\alpha < \lambda$  there is a group  $G$  whose tower has height  $\alpha$ , but for any non-zero  $\beta < \lambda$  there is a forcing extension in which the height of the tower of  $G$  has height  $\beta$ .*

I would like for most of the remaining time to give a stratospheric view of the proof of this theorem. For the details, which are abundant, I refer you to our paper [1997].

### The set-theoretic kernel

Perhaps the key set-theoretic idea is the realization that forcing can make non-isomorphic structures isomorphic. We begin with the following problem.

**Warm-up Problem.** Construct rigid non-isomorphic objects  $S$  and  $T$  which can be made isomorphic by forcing while remaining rigid.

The solution is to use generic Souslin trees. If you add mutually generic normal Souslin trees, by forcing with normal  $\alpha$ -trees ( $\alpha < \omega_1$ ) ordered by end-extension, the resulting  $\omega_1$ -trees  $S$  and  $T$  will be rigid and non-isomorphic. To see why this is so, suppose, for example, that  $\dot{\pi}$  is the name of a purported isomorphism between  $S$  and  $T$ . A simple bootstrap argument provides conditions  $S_\alpha$  and  $T_\alpha$  which decide  $\dot{\pi} \restriction S_\alpha$ , and by strategically extending some paths in  $S_\alpha$  but not the image of the paths in  $T_\alpha$ , one obtains a stronger condition  $\langle S_{\alpha+1}, T_{\alpha+1} \rangle$  which forces that  $\dot{\pi}$  is not an isomorphism, a contradiction. A similar bootstrap argument shows that  $S$  and  $T$ , individually, are rigid. Since by a back-and-forth argument any two normal  $\alpha$ -trees, for  $\alpha < \omega_1$ , are isomorphic, there are abundant partial isomorphisms on the initial segments of  $S$  and  $T$ , and by forcing with these partial isomorphisms, one adds by forcing an isomorphism between  $S$  and  $T$ . The combined forcing can be viewed as forcing with normal  $\alpha$ -trees  $S_\alpha$  and  $T_\alpha$  and an isomorphism between them, and consequently the bootstrap argument can be modified to show that the isomorphism which is added between  $S$  and  $T$  is unique. That is, the trees  $S$  and  $T$  remain rigid even after they are forced to be isomorphic, as desired.

More generally, by an Easton support forcing iteration, we obtain such objects simultaneously for every regular cardinal:

**Theorem.** (HAMKINS AND THOMAS 1997) *One can add, for every regular cardinal  $\kappa$ , a set  $\{T_\alpha \mid \alpha < \kappa\}$  of pair-wise non-isomorphic rigid trees such that for any equivalence relation  $E$  on  $\kappa^+$ , there is a forcing extension preserving rigidity in which the isomorphism relation on the trees is exactly  $E$ .*

By this theorem, therefore, we have a collection of non-isomorphic rigid objects

which we can make isomorphic at will, while preserving their rigidity. Later, these objects will be the unit elements in elaborate graphs whose automorphism groups we want to control precisely by forcing.

### The algebraic kernel

We begin with the admission that automorphism towers are too difficult to handle, and so instead we work with the *normalizer tower* of a group. Given  $G \leq H$  define

$$\begin{aligned} N_0(G) &= G \\ N_{\alpha+1}(G) &= N_H(N_\alpha(G)) \\ N_\lambda(G) &= \cup_{\alpha < \lambda} N_\alpha(G), \text{ if } \lambda \text{ is a limit} \end{aligned}$$

The reason for doing so lies in the following amazing fact:

**Fact.** *The automorphism tower of a centerless group  $G$  is exactly the normalizer tower of  $G$  computed in the terminal group  $H = G_\gamma$ . That is,  $G_\alpha = N_\alpha(G)$ .*

**Fact.** *Conversely, if  $H \leq \text{Aut}(K)$ , where  $K$  is a field, then by making a few modifications (adding a few bells and whistles) to the normalizer tower of  $H$  in  $\text{Aut}(K)$ , one obtains an automorphism tower of the same height.*

Thus, to make automorphism towers of a specific height, we need only make normalizer towers of that height in the automorphism group of a field. This latter limitation is considerably loosened in light of the following theorem of Fried and Kollar:

**Theorem.** (FRIED AND KOLLAR 1981) *By adding points, any graph can be made into a field with the same automorphism group.*

Thus, we can restrict our attention to subgroups of the automorphism groups of graphs. Since any tree can be represented as a graph, we make the connection with our earlier set-theoretic argument. There, we obtained a delicate skill to make trees isomorphic while preserving their rigidity. By combining these trees in elaborate combinations, we will construct graphs whose automorphism groups we can precisely modify by forcing. Let us see how this is done.

In order to construct the tall normalizer towers, we take as a unit some rigid graph, which I will denote by  $\triangle$ , and build the following large graph and the corresponding subgroup of its automorphism group:

$$\triangle \triangle \boxed{\triangle \triangle} \boxed{\boxed{\triangle \triangle} \boxed{\triangle \triangle}} \cdots \text{up to } \alpha$$

The intended subgroup, a large wreath product, is indicated by the boxes. The idea is that while the full automorphism group can freely permute the  $\triangle$ s, the subgroup

we are interested in consists of those permutations which iteratively swap the two components of any of the boxes. Thus, every element of the subgroup fixes the first two  $\triangle$ s, but there is, for example, a group element which swaps the third and the fourth  $\triangle$  and swaps the fifth and the sixth  $\triangle$  for the eighth and the seventh  $\triangle$ , respectively. The subgroup is therefore simply a large wreath product.

The normalizer tower of this group in the full automorphism group of the graph can be iteratively computed with ease. At the first step, one must add the permutations which swap the first two triangles, leading to:

$$\boxed{\triangle \triangle} \boxed{\triangle \triangle} \boxed{\boxed{\triangle \triangle} \boxed{\triangle \triangle}} \dots \text{ up to } \alpha$$

At the next stage one adds the permutations which swap the first two boxes here, producing:

$$\boxed{\boxed{\triangle \triangle} \boxed{\triangle \triangle}} \boxed{\boxed{\triangle \triangle} \boxed{\triangle \triangle}} \dots \text{ up to } \alpha$$

Inductively, one sees that the normalizer tower continues to grow for  $\alpha$  many steps and then terminates. That is, the normalizer tower of the initial group inside the full automorphism group has height  $\alpha$ . By adding the bells and whistles I mentioned earlier, then, we have constructed a centerless group whose automorphism tower has height  $\alpha$ .

Now we come to the exciting twist, which is to use the rigid pairwise non-isomorphic objects obtained from the set-theoretic part of the argument. Representing these various objects by  $\triangle$ ,  $\square$ ,  $\bigcirc$  and  $\diamond$  we build a subgroup of the following graph:

$$\triangle \square \boxed{\bigcirc \bigcirc} \boxed{\boxed{\diamond \diamond} \boxed{\diamond \diamond}} \dots$$

The key point is now that by forcing the objects to be isomorphic  $\triangle \cong \square \cong \bigcirc \cong \diamond \cong \dots$  up to  $\beta$ , one thereby transforms the previous graph to look like the original picture

$$\triangle \triangle \boxed{\triangle \triangle} \boxed{\boxed{\triangle \triangle} \boxed{\triangle \triangle}} \dots$$

for  $\beta$  many steps. After the forcing, therefore, the normalizer tower will have height  $\beta$ , as desired.

One can also make the height of the tower go down by activating a sort of wall which prevents the normalizer tower proceeding through:

$$\begin{array}{c} \triangle \triangle \boxed{\triangle \triangle} \boxed{\boxed{\triangle \triangle} \boxed{\triangle \triangle}} \dots \\ \boxed{\boxed{\square \square} \boxed{\square \square}} \\ \boxed{\boxed{\square \square} \boxed{\square \square}} \end{array}$$



By forcing  $\square \cong \triangle$ , it is easy to see that the normalizer tower will not continue past the wall.

In summary, the height of the normalizer tower can be completely controlled by the forcing which makes certain trees isomorphic while retaining their rigidity. The height of the corresponding automorphism tower, therefore, can also be completely controlled, and the proof of the theorem is complete.

Let me conclude with a discussion of the state of the art with respect to the heights of automorphism towers. In the case of centerless groups, we have Simon Thomas' theorem that the automorphism tower of a group of size  $\kappa$  terminates in fewer than  $(2^\kappa)^+$  many steps. Of course, this bound cannot be the best possible bound, because there are only  $2^\kappa$  many groups of size  $\kappa$ , and so the actual supremum of the towers is strictly below  $(2^\kappa)^+$ . Furthermore, by forcing it is easy to make  $2^\kappa$  as large as desired, and so the "bound" can be pushed higher and higher. Nevertheless, Thomas' theorem is optimal in the sense that no better upper bound will ever be found:

**Theorem.** (JUST, THOMAS, SHELAH 1997) *Suppose that the GCH holds, that  $\kappa$  is a regular uncountable cardinal, that  $\text{cof}(\lambda) > \kappa$  and that  $\alpha < \lambda^+$ . Then there is a cardinal-preserving forcing extension in which  $2^\kappa = \lambda$  and there is a group  $G$  of size  $\kappa$  whose automorphism tower has height  $\alpha$ .*

Missing from this analysis are the countable groups. How tall is the automorphism tower of a centerless countable group?

In the arbitrary case, where non-centerless groups are included, the best uniform upper bound on the height of the automorphism tower of a group is. . . . .the next inaccessible cardinal. Actually, it is easy to improve slightly on this, by taking the next cardinal  $\delta$  above the size of the group such that  $V_\delta$  is a model of ZFC or at least a sufficiently powerful set theory to prove the Main Theorem above. One has the sense, though, that this is not the right answer.

Such is our pitiful knowledge even in the case of finite groups! The most we know about an upper bound for the height of the automorphism tower of a finite group is something like the next inaccessible cardinal. Contrast this with the fact that the tallest towers we know of for finite groups have height  $\omega + n$  for finite  $n$ . So the true answer lies somewhere between  $\omega + n$  and the least inaccessible cardinal. How tall is the automorphism tower of a finite group?

The following questions, to my knowledge, remain open:

**Question.** *Is there a countable group with an uncountable automorphism tower?*

**Question.** *Is there a finite group with an uncountable automorphism tower?*

**Question.** *Is there a finite group  $G$  such that  $G_\omega$  is infinite?*

**Question.** *For which ordinals  $\gamma$  is there a group whose tower becomes centerless in exactly  $\gamma$  many steps?*

**Question.** *Is there a group  $G$  whose automorphism tower has height  $(2^{|G|})^+$  or more?*

And finally, let me state Scott's question, unanswered for 35 years:

**Question.** (SCOTT 1964) *Is the finite part of the automorphism tower of every finite group eventually periodic?*

Perhaps Scott's question is peculiar, because no nontrivial instances of this periodicity phenomenon are known.

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